Among electromechanical systems Routh's theorem covers systems with superconductive loops. We disregard those exceptional cases when both the quadratic forms mentioned above vanish for one and the same $v_1, \ldots, v_{n-m}$. Then the preceding discussion signifies that the forms of equilibrium under the action of a magnetic field which are stable when the field is created by loops with finite conductivity, are stable also for superconductivity, but forms exist which are stable only in the case of superconductive loops. Systems with superconductive loops possess, consequently, qualitative singularities in the "purely mechanical" sense being considered here.

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A PRACTICAL METHOD FOR COMPUTING NORMAL FORMS IN NONLINEAR OSCILLATION PROBLEMS

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The applications of normal forms (see [1] for the history and a bibliography) to nonlinear oscillations have been outlined in [2]. As one of the applied problems we indicate the investigations of Ishlinskii ([3], Appendix 2) in [4]. One unsolved problem that remains is the derivation of recurrence formulas for computing the coefficients of the normalizing transformations and of the normal forms. These formulas have been derived below for a general case in the theory of oscillations (the absence of nonprime elementary divisors in the matrix of the linear part) on the basis of Briuno's theorem [1].

1. Statement of the problem. Let an oscillatory system be described by an nth-order autonomous system of differential equations, in which the variables can also be complex valued. We assume that the elementary divisors of the matrix of its linear part are prime. For oscillatory systems with Hermitian or unitary matrices of the linear part the latter condition is fulfilled by virtue of the Weierstrass theorem (for example, see [5], Sect. I.1.14). We shall assume that the original system has already been reduced to diagonal form and that its right-hand side is analytic in some neighborhood of the null values with, in general, complex coefficients

$$\frac{dx_\nu}{dt} = \lambda_\nu x_\nu + \sum_{\varkappa=2}^{\infty} \sum a^\nu_{j_1\ldots j_\varkappa} x_{j_1}\ldots x_{j_\varkappa} \quad (\nu = 1, \ldots, n) \tag{1.1}$$

The vector $\Lambda = (\lambda_1, \ldots, \lambda_n)$ is assumed nonzero, i.e. as having at least one nonzero component. The coefficients are assumed to be symmetrized, i.e.

$$a^\nu_{j_2 j_1} = a^\nu_{j_1 j_2}, \quad a^\nu_{\{j_1\ldots j_\varkappa\}} = \text{idem} \quad (\varkappa = 3, 4, \ldots; \ \nu = 1, \ldots, n)$$

and everywhere $\{\alpha\beta\ldots\omega\}$ denotes any permutation of the positive integers $\alpha, \beta, \ldots, \omega$. In (1.1) and everywhere without so specifying the summation over twice-occurring indices, taking the values $1, 2, \ldots, n$ independently of each other (by virtue of the symmetry of the coefficients).

By Briuno's theorem ([1], Sect. 0, Para. II and Chap. I, Sect. 1, Para. I) there exists an invertible (but, in general, nonunique and, in some cases, divergent) normalizing transformation with, in general, complex coefficients (we represent it again in a symmetrized form)

$$x_\nu = y_\nu + \sum_{\varkappa=2}^{\infty} \alpha^\nu_{j_1\ldots j_\varkappa} y_{j_1}\ldots y_{j_\varkappa} \quad (\nu = 1, \ldots, n) \tag{1.2}$$

$$(\alpha^\nu_{j_2 j_1} = \alpha^\nu_{j_1 j_2}, \ \alpha^\nu_{\{j_1\ldots j_\varkappa\}} = \text{idem}; \ \varkappa = 3, 4, \ldots; \ \nu = 1, \ldots, n)$$

taking system (1.1) to the normal form

$$\frac{dy_\nu}{dt} = \lambda_\nu y_\nu + y_\nu \sum_{(\Lambda\cdot Q)=0} g_{\nu Q} y_1^{q_1}\ldots y_n^{q_n} \quad (\nu = 1, \ldots, n) \tag{1.3}$$

Here $Q = (q_1, \ldots, q_n)$ is a vector with integral components, moreover,

$$q_\nu \geqslant -1, \quad q_j \geqslant 0 \quad (j \neq \nu) \quad (\nu, j = 1, \ldots, n) \tag{1.4}$$

and $g_{\nu Q}$ are nonsymmetrized coefficients of the normal form. The summation in (1.3) takes place only over the resonance terms satisfying the resonance equation

$$(\Lambda \cdot Q) \equiv \lambda_1 q_1 + \ldots + \lambda_n q_n = 0 \tag{1.5}$$

Let us symmetrize the coefficients of the normal form (1.3) and write it as

$$\frac{dy_\nu}{dt} = \lambda_\nu y_\nu + \sum_{\varkappa=2}^{\infty} \sum \varphi^\nu_{j_1\ldots j_\varkappa} y_{j_1}\ldots y_{j_\varkappa} \quad (\nu = 1, \ldots, n) \tag{1.6}$$

$$(\varphi^\nu_{j_2 j_1} = \varphi^\nu_{j_1 j_2}, \ \varphi^\nu_{\{j_1\ldots j_\varkappa\}} = \text{idem}; \ \varkappa = 3, 4, \ldots; \ \nu = 1, \ldots, n)$$

It is understood that the nonzero coefficients $\varphi^\nu_{j_1\ldots j_\varkappa}$ in (1.6) are determined by representation (1.3).

2. Fundamental identities. Substituting (1.2) into (1.1) we obtain, by virtue of (1.3), the following formal identities:

$$\sum \varphi^\nu_{j_1 j_2} y_{j_1} y_{j_2} + \ldots + \sum \varphi^\nu_{j_1\ldots j_\varkappa} y_{j_1}\ldots y_{j_\varkappa} + \ldots + \sum \alpha^\nu_{j_1 j_2} (y_{j_1}^\cdot y_{j_2} + y_{j_1} y_{j_2}^\cdot) + \ldots$$
$$\sum \alpha^\nu_{j_1\ldots j_\varkappa} (y^\cdot_{j_1} y_{j_2}\ldots y_{j_\varkappa} + \ldots + y_{j_1}\ldots y_{j_{\mu-1}} y_{j_\mu}^\cdot y_{j_{\mu+1}}\ldots y_{j_\varkappa} + \ldots$$
$$y_{j_1}\ldots y_{j_{\varkappa-1}} y_{j_\varkappa}^\cdot) + \ldots = \lambda_\nu \sum \alpha^\nu_{j_1 j_2} y_{j_1} y_{j_2} + \ldots + \lambda_\nu \sum \alpha^\nu_{j_1\ldots j_\varkappa} y_{j_1}\ldots y_{j_\varkappa} + \ldots$$

$$\sum a^{\nu}_{j_1j_2}\Big(\sum \alpha^{j_1}_{j_1^1}y_{j_1^1}+\ldots+\sum \alpha^{j_1}_{j_1^1\ldots j_{\vartheta}^1}y_{j_1^1}\ldots y_{j_{\vartheta}^1}+\ldots+$$
$$\sum \alpha^{j_1}_{j_1^1\ldots j^1_{k-1}}y_{j_1^1}\ldots y_{j^1_{k-1}}+\ldots\Big)\Big(\sum \alpha^{j_2}_{j_1^2}y_{j_1^2}+\ldots+$$
$$\sum \alpha^{j_2}_{j_1^2\ldots j_{\vartheta}^2}y_{j_1^2}\ldots y_{j_{\vartheta}^2}+\ldots+\sum \alpha^{j_2}_{j_1^2\ldots j^2_{k-1}}y_{j_1^2}\ldots y_{j^2_{k-1}}+\ldots)+\ldots+$$
$$\sum a^{\nu}_{j_1\ldots j_\varkappa}\Big(\sum \alpha^{j_1}_{j_1^1}y_{j_1^1}+\ldots+\sum \alpha^{j_1}_{j_1^1\ldots j_{\vartheta}^1}y_{j_1^1}\ldots y_{j_{\vartheta}^1}+\ldots+$$
$$\sum \alpha^{j_1}_{j_1^1\ldots j^1_{k-\varkappa+1}}y_{j_1^1}\ldots y_{j^1_{k-\varkappa+1}}+\ldots\Big)\ldots\Big(\sum \alpha^{j_\varkappa}_{j_1^\varkappa}y_{j_1^\varkappa}+\ldots+$$
$$\sum \alpha^{j_\varkappa}_{j_1^\varkappa\ldots j_{\vartheta}^\varkappa}y_{j_1^\varkappa}\ldots y_{j_{\vartheta}^\varkappa}+\ldots+\sum \alpha^{j_\varkappa}_{j_1^\varkappa\ldots j^\varkappa_{k-\varkappa+1}}y_{j_1^\varkappa}\ldots y_{j^\varkappa_{k-\varkappa+1}}+\ldots\Big)+\ldots$$
$$(\nu=1,\ldots,n)$$

Here and below, in correspondence with (1.2) $\alpha^j_h=\delta_{jh}$ $(j,h=1,\ldots,n)$ (δ_{jh} is the Kronecker symbol).

Taking (1.6) into account we write out the terms with the kth powers of the variables in these identities

$$\sum \varphi^{\nu}_{j_1\ldots j_k}y_{j_1}\ldots y_{j_k}+\sum_{\varkappa=2}^{k-1}\sum_{\mu=1}^{\varkappa}\sum_{j_1,\ldots,j_\varkappa}\alpha^{\nu}_{j_1\ldots j_\varkappa}y_{j_1}\ldots y_{j_{\mu-1}}y_{j_{\mu+1}}\ldots$$
$$y_{j_\varkappa}\sum_{j_1^\mu,\ldots,j^\mu_{k-\varkappa+1}}\varphi^{j_\mu}_{j_1^\mu\ldots j^\mu_{k-\varkappa+1}}y_{j_1^\mu}\ldots y_{j^\mu_{k-\varkappa+1}}+$$
$$\sum(\lambda_{j_1}+\ldots+\lambda_{j_k})\alpha^{\nu}_{j_1\ldots j_k}y_{j_1}\ldots y_{j_k}=\lambda_\nu\sum \alpha^{\nu}_{j_1\ldots j_k}y_{j_1}\ldots y_{j_k}+$$
$$\sum_{\varkappa=2}^{k-1}\sum_{i_1,\ldots,i_\varkappa}a^{\nu}_{i_1\ldots i_\varkappa}\sum_{\mu_1+\ldots+\mu_\varkappa=k}\sum_{j_1^1,\ldots,j^\varkappa_{\mu_\varkappa}}\alpha^{i_1}_{j_1^1\ldots j^1_{\mu_1}}\ldots\alpha^{i_\varkappa}_{j_1^\varkappa\ldots j^\varkappa_{\mu_\varkappa}}\times$$
$$y_{j_1^1}\ldots y_{j^1_{\mu_1}}\ldots y_{j_1^\varkappa}\ldots y_{j^\varkappa_{\mu_\varkappa}}+\sum a^{\nu}_{j_1\ldots j_k}y_{j_1}\ldots y_{j_k}\qquad(\nu=1,\ldots,n)\tag{2.1}$$

Here $\mu_1,\ldots,\mu_{k-1}$ are positive integers. Let us compare the coefficients of $y_{j_1}\ldots y_{j_k}$, where $j_1,\ldots,j_k$ is any fixed sequence of positive integers not exceeding n. Nonsymmetric coefficients generated during the computations must be symmetrized because the coefficients $\alpha^{\nu}_{j_1\ldots j_k}$ and $\varphi^{\nu}_{j_1\ldots j_k}$ to be determined are subject to this condition.

In identities (2.1) in the second term on the left in every summand of the sum from $\mu=1$ to $\varkappa$ we replace the summation indices in the following way: $j_1,\ldots,j_{\mu-1}$, $j_{\mu+1},\ldots,j_\varkappa$ by $i_1,\ldots,i_{\varkappa-1}$, respectively; index j_μ by i and the indices $j_1^\mu,\ldots,j^\mu_{k-\varkappa-1}$ by $i_\varkappa,i_{\varkappa+1},\ldots,i_k$, respectively. It is obvious that all the additive sums from $\mu=1$ to $\varkappa$ are like and, therefore, we represent them as one of the additive sums taken $\varkappa$ times. To symmetrize the latter we examine all combinations $p_1,\ldots,p_{\varkappa-1}$ of the $\varkappa-1$ positive integers from $1,\ldots,k$ (we denote their number by $C_k^{\varkappa-1}$). Finally, we denote the summation indices $i_{p_1},\ldots,i_{p_{\varkappa-1}}$ by $j_{p_1},\ldots,j_{p_{\varkappa-1}}$, while the rest of the indices $i_1,\ldots,i_k$ by $j_\varkappa',j_{\varkappa+1},\ldots,j_k'$.

We have thus carried out the transformations

$$\sum_{=1}^{\varkappa} \sum_{j_1,\ldots,j_\varkappa,j_1^\mu,\ldots,j_{k-\varkappa+1}^\mu} \alpha^{\nu}_{j_1\ldots j_\varkappa} \varphi^{j_\mu}_{j_1^\mu\ldots j_{k-\varkappa+1}^\mu} y_{j_1}\ldots y_{j_{\mu-1}} y_{j_{\mu+1}}\ldots y_{j_\varkappa} y_{j_1^\mu}\ldots y_{j^\mu_{k-\varkappa+1}} =$$

$$\varkappa \sum_{i_1,\ldots,i_k,i} \alpha^{\nu}_{i_1\ldots i_{\varkappa-1}i}\, \varphi^{i}_{i_\varkappa\ldots i_k} y_{i_1}\ldots y_{i_k} = \varkappa \sum_{=1} \sum_{j_1,\ldots,j_k} [C_k^{\varkappa-1}]^{-1} \times$$

$$S^{p_1,\ldots,p_{\varkappa-1}}_{1,\ldots,k}\, \alpha^{\nu}_{j_{p[1]}\ldots j_{p[\varkappa-1]}i}\, \varphi^{i}_{j_\varkappa'\ldots j_k'} y_{j_1}\ldots y_{j_k} \qquad (\nu = 1,\ldots,n) \qquad (2.2)$$

where $p[1] \equiv p_1, \ldots, p[\varkappa-1] \equiv p_{\varkappa-1}$. Here $S^{p_1,\ldots,p_{\varkappa-1}}_{1,\ldots,k}$ denotes summation over all combinations of $\varkappa - 1$ positive numbers from $1, \ldots, k$. We remark that the numbers $j_{p_1}, \ldots, j_{p_{\varkappa-1}}$ can be (even all of them) like, because they (as also $j_\varkappa'$, $j_{\varkappa+1}', \ldots, j_k'$) range during the summation over the values $1, \ldots, n$ independently of each other. However, as regards the indices on i or j, they are all distinct, and that is why the combinations are a type of couplings.

Let us tranform the second term on the right in (2.1). We replace the summation indices $j_1^1, \ldots, j_{\mu_1}^1, \ldots, j_1^\varkappa, \ldots, j_{\mu_\varkappa}^\varkappa$ $(\mu_1 + \ldots + \mu_\varkappa = k)$ by $j_1, \ldots, j_k$. To symmetrize the coefficient of $y_{j_1}\ldots y_{j_k}$ we consider all combinations $p_1, \ldots, p_{\mu_1}$ of the μ_1 positive integers from $1, \ldots, k$ (we denote their number by $C_k^{\mu_1}$), next all combinations $p_{\mu+1}, \ldots, p_{\mu_1+\mu_2}$ of the μ_2 positive integers $1, \ldots, k \setminus p_1, \ldots, p_{\mu_1}$ (we denote tneir number by $C^{\mu_2}_{k-\mu_1}$), etc., all the way up to the combinations $p_{k-\mu_\varkappa-\mu_{\varkappa-1}+1}, \ldots, p_{k-\mu_\varkappa}$ of $\mu_{\varkappa-1}$ from the remaining $\mu_{\varkappa-1} + \mu_\varkappa$ positive integers $1, \ldots, k \setminus p_1, \ldots, p_{\mu_1}, p_{\mu_1+1}, \ldots, p_{k-\mu_\varkappa-\mu_{\varkappa-1}}$ (we denote their number by $C^{\mu_{\varkappa-1}}_{\mu_{\varkappa-1}+\mu_\varkappa}$). Thus

$$\sum_{j_1^1,\ldots,j^\varkappa_{\mu_\varkappa}} \alpha^{i_1}_{j_1^1\ldots j^1_{\mu_1}} \ldots \alpha^{i_\varkappa}_{j_1^\varkappa\ldots j^\varkappa_{\mu_\varkappa}} y_{j_1^1}\ldots y_{j^1_{\mu_1}} \ldots y_{j_1^\varkappa}\ldots y_{j^\varkappa_{\mu_\varkappa}} =$$

$$\sum_{j_1,\ldots,j_k} [C_k^{\mu_1} C^{\mu_2}_{k-\mu_1} \ldots C^{\mu_{\varkappa-1}}_{\mu_{\varkappa-1}+\mu_\varkappa}]^{-1} S^{p[k-\mu(\varkappa)-\mu(\varkappa-1)+1],\ldots,p[k-\mu(\varkappa)]}_{1,\ldots,k\setminus p[1],\ldots,p[k-\mu(\varkappa)-\mu(\varkappa-1)]} \ldots$$

$$\ldots S^{p[\mu(1)+1],\ldots,p[\mu(1)+\mu(2)]}_{1,\ldots,k\setminus p[1],\ldots p[\mu(1)]} S^{p[1],\ldots,p[\mu(1)]}_{1,\ldots,k} \alpha^{i_1}_{j_{p[1]}\ldots j_{p[\mu(1)]}} \alpha^{i_2}_{j_{p[\mu(1)+1]}\ldots j_{p[\mu(1)+\mu(2)]}} \ldots$$

$$\ldots \alpha^{i_{\varkappa-1}}_{j_{p[k-\mu(\varkappa)-\mu(\varkappa-1)+1]}\ldots j_{p[k-\mu(\varkappa)]}} \alpha^{i_\varkappa}_{j_{p[k-\mu(\varkappa)+1]}\ldots j_{p[k]}} y_{j_1}\ldots y_{j_k} \qquad (2.3)$$

where here and below we have denoted $\mu(\varkappa) \equiv \mu_\varkappa$, $p[m] \equiv p_m$. Here $S^{p[1],\ldots,p[\mu(1)]}_{1,\ldots,k}$ denotes summation over all combinations of the μ_1 positive numbers $p_1, \ldots, p_{\mu_1}$ from $1, \ldots, k$; $S^{p[\mu(1)+1],\ldots,p[\mu(1)+\mu(2)]}_{1,\ldots,k\setminus p[1],\ldots,p[\mu(1)]}$ denotes summation over all combinations of the μ_2 positive integers $p_{\mu_1+1}, \ldots, p_{\mu_1+\mu_2}$ from the remaining $k - \mu_1$ positive integers $1, \ldots, k \setminus p_1, \ldots, p_{\mu_1}$, etc.

Now, using (2.2) and (2.3) we write identities (2.1) in the symmetrized form

$$\sum \varphi^{\nu}_{j_1\ldots j_k} y_{j_1}\ldots y_{j_k} + \sum_{\varkappa=2}^{k-1} \sum (2.2) + \sum (\lambda_{j_1} + \ldots + \lambda_{j_k} - \lambda_\nu)\, \alpha^{\nu}_{j_1\ldots j_k} y_{j_1}\ldots y_{j_k} =$$

$$\sum a^{\nu}_{j_1\ldots j_k} y_{j_1}\ldots y_{j_k} + \sum_{\varkappa=2}^{k-1} \sum_{i_1,\ldots,i_\varkappa=1}^{n} a^{\nu}_{i_1\ldots i_\varkappa} \sum_{\mu_1+\ldots+\mu_\varkappa=k} \sum (2.3) \qquad (2.4)$$

Here, for brevity, Σ (2.2) and Σ (2.3) denote the entire right-hand side of the last equality in (2.2) and the entire right-hand side of (2.3), respectively.

3. Computational alternative. We introduce the symbol

$$\Delta^{\nu}_{j_1\ldots j_k} = \begin{cases} 1, & \text{if} \quad \lambda_\nu = \lambda_{j_1} + \ldots + \lambda_{j_k} \\ 0, & \text{if} \quad \lambda_\nu \neq \lambda_{j_1} + \ldots + \lambda_{j_k} \end{cases} \qquad (3.1)$$
$$(\nu, j_1, \ldots, j_k = 1, \ldots, n)$$

The following alternative holds.

1) Let the values of ν, j_1, . . ., j_k (and of the real parameters of the original oscillatory system, on which λ_ν, λ_{j_1}, . . ., λ_{j_k} depend) be such that the parentheses in the last sum in the left-hand side of identities (2.4) is nonzero, i.e. $\Delta^{\nu}_{j_1\ldots j_k} = 0$. By equating the terms with $y_{j_1} \ldots y_{j_k}$ in the left- and right-hand sides of identities (2.4), we note that under the assumption made, such a term is automatically absent from the first sum on the left. Indeed, returning to representation (1.3), we write this term in the form

$$y_\nu \varphi^{\nu}_{j_1\ldots j_k} y_{j_1} \ldots y_{j_k} y_\nu^{-1}$$

For this term $(\Lambda \cdot Q) = \lambda_{j_1} \cdot 1 + \ldots + \lambda_{j_k} \cdot 1 + \lambda_\nu \cdot (-1) \neq 0$, while according to representation (1.3) only those terms occur in the first sum on the left in (2.4) for which $(\Lambda \cdot Q) = 0$. By equating in identities (2.4) the coefficients of $y_{j_1} \ldots y_{j_k}$, we obtain a formula for the coefficients of the normalizing transformation (1.2)

$$\alpha^{\nu}_{j_1\ldots j_k} = \frac{1 - \Delta^{\nu}_{j_1\ldots j_k}}{\lambda_{j_1} + \ldots + \lambda_{j_k} - \lambda_\nu} B^{\nu}_{j_1\ldots j_k} \qquad (3.2)$$
$$(\lambda_{j_1} + \ldots + \lambda_{j_k} - \lambda_\nu \neq 0; \quad \nu, j_1, \ldots, j_k = 1, \ldots, n)$$

$$B^{\nu}_{j_1\ldots j_k} = a^{\nu}_{j_1\ldots j_k} + \sum_{\varkappa=2}^{k-1} \Bigg\{ \sum_{i_1,\ldots,i_\varkappa=1}^{n} a^{\nu}_{i_1\ldots j_\varkappa} \sum_{\mu_1+\ldots+\mu_\varkappa=k} [C_k^{\mu_1} C_{k-\mu_1}^{\mu_2} \ldots C^{\mu_{\varkappa-1}}_{\mu_{\varkappa-1}+\mu_\varkappa}]^{-1} \times$$
$$S^{p[k-\mu(\varkappa)-\mu(\varkappa-1)+1],\ldots,\, p[k-\mu(\varkappa)]}_{1,\ldots,\, k \setminus p[1],\ldots,\, p[k-\mu(\varkappa)-\mu(\varkappa-1)]} \ldots S^{p[\mu(1)+1],\ldots,\, p[\mu(1)+\mu(2)]}_{1,\ldots,\, k \setminus p[1],\ldots,\, p[\mu(1)]} S^{p[1],\ldots,\, p[\mu(1)]}_{1,\ldots,\, k} \times$$
$$\alpha^{i_1}_{j_{p[1]}\cdots j_{p[\mu(1)]}} \alpha^{i_2}_{j_{p[\mu(1)+1]}\cdots j_{p[\mu(1)+\mu(2)]}} \ldots \alpha^{i_{\varkappa-1}}_{j_{p[k-\mu(\varkappa)-\mu(\varkappa-1)+1]} \cdots j_{p[k-\mu(\varkappa)]}} \times$$
$$\alpha^{i_\varkappa}_{j_{p[k-\mu(\varkappa)+1]}\cdots j_{p[k]}} - \varkappa\, [C_k^{\varkappa-1}]^{-1} S^{p_1,\ldots,p_{\varkappa-1}}_{1,\ldots,\, k} \sum_{i=1}^{n} \alpha^{\nu}_{j_{p[1]}\cdots j_{p[\varkappa-1]}} \varphi^{i}_{j_{\varkappa'}\ldots,j_{k'}} \Bigg\} \qquad (3.3)$$
$$(\nu, j_1, \ldots, j_k = 1, \ldots, n)$$

2) Let us assume that the values of ν, j_1, . . ., j_k are such that the parentheses in the last sum on the left-hand side of identities (2.4) equals zero, i.e. $\Delta^{\nu}_{j_1\ldots j_k} = 1$. This signifies, firstly, that the quantity $\alpha^{\nu}_{j_1\ldots j_k}$ can be chosen arbitrarily, in particular, equal to zero or defined by continuity from the values of the real parameters. Secondly, by equating the terms with $y_{j_1} \ldots y_{j_k}$ in the left and right sides of identities (2.4), we now obtain a formula for the symmetrized coefficients of the normal form (1.6)

$$\varphi^{\nu}_{j_1\ldots j_k} = \Delta^{\nu}_{j_1\ldots j_k} B^{\nu}_{j_1\ldots j_k} \quad (\nu, j_1, \ldots, j_k = 1, \ldots, n) \qquad (3.4)$$

where $\Delta^{\nu}_{j_1\ldots j_k}$ is given by (3.1), while $B^{\nu}_{j_1\ldots j_k}$ by (3.3).

Notes. 1°. In formulas (3.2) and (3.4) expression $\Delta^{\nu}_{j_1\ldots j_k}$ is intended to serve as a warning. In fact, according to formula (3.4) for $\Delta^{\nu}_{j_1\ldots j_k} = 0$ we have $\varphi^{\nu}_{j_1\ldots j_k} = 0$ (Case (1)). For $\Delta^{\nu}_{j_1\ldots j_k} = 1$ $(\lambda_{j_1} + \ldots + \lambda_{j_k} - \lambda_\nu = 0)$ the fraction preceding brackets (sic)(*) (see footnote on the next page) in formula (3.2) loses its meaning, since it is then indeterminate. We recall that in this case the value of $\alpha^{\nu}_{j_1\dots j_k}$ can be selected arbitrarily.

2°. Let the indices $j_1,\dots,j_k$ be arranged so that the first χ of them $(1 \leqslant \chi \leqslant k)$ are distinct and let j_1 occur m_{j_1} times,..., j_χ occur m_{j_χ} times $(m_{j_1} + \dots + m_{j_\chi} = k)$. The number of different permutations of these indices is

$$N = \frac{k!}{m_{j_1}! \dots m_{j_\chi}!}$$

This means that in the sum

$$\sum_{j_1,\dots,j_k=1}^{n} a^{\nu}_{j_1\dots j_k} x_{j_1} \dots x_{j_k}$$

There are in all N similar terms containing $x_{j_1}\dots x_{j_k}$. Therefore, N also is the factor in the passage from the symmetrized coefficients to the ordinary ones, i.e. when all the monomials in the sum are distinct.

3°. We refer the reader to Briuno's article [1] for questions on the convergence of the normalizing transformations.

4. Formulas for the coefficients of quadratic and cubic terms. For $k = 2$ formulas (3.2) – (3.4) yield the symmetrized coefficients of the quadratic terms of the normalizing transformation (1.2)

$$\alpha^{\nu}_{j_1j_2} = \frac{a^{\nu}_{j_1j_2}}{\lambda_{j_1} + \lambda_{j_2} - \lambda_\nu} \qquad (\lambda_{j_1} + \lambda_{j_2}^{*} - \lambda_\nu \neq 0;\ \nu, j_1, j_2 = 1, \dots, n) \qquad (4.1)$$

and of the normal form (1.6)

$$\varphi^{\nu}_{j_1j_2} = a^{\nu}_{j_1j_2} \quad (\lambda_{j_1} + \lambda_{j_2} - \lambda_\nu = 0;\ \ \nu, j_1, j_2 = 1, \dots, n) \qquad (4.2)$$

Here $a^{\nu}_{j_1j_2}$ are the symmetrized quadratic coefficients in (1.1) ($\lambda_1, \dots, \lambda_n$, see this expression) We emphasize that by the definition of a normal form $\varphi^{\nu}_{j_1j_2} = 0$ for those values of ν, j_1, j_2 taken from $1, \dots, n$ for which $\lambda_{j_1} + \lambda_{j_2} - \lambda_\nu \neq 0$. On the other hand, when $\lambda_{j_1} + \lambda_{j_2} - \lambda_\nu = 0$ the coefficients $\alpha_{j_1j_2}{}^{\nu}$ can be chosen arbitrarily.

For the cubic coefficients in (1.2) and (1.6), from formulas (3.2) – (3.4) with $k = 3$ we have

$$\alpha^{\nu}_{j_1j_2j_3} = \frac{1}{\lambda_{j_1} + \lambda_{j_2} + \lambda_{j_3} - \lambda_\nu} \Big\{a^{\nu}_{j_1j_2j_3} + \frac{2}{3} \sum_{i=1}^{n} [a^{\nu}_{j_1i}\alpha^{i}_{j_2j_3} + a^{\nu}_{j_2i}\alpha^{i}_{j_3j_1} + a^{\nu}_{j_3i}\alpha^{i}_{j_1j_2} - (\alpha^{\nu}_{j_1i}\varphi^{i}_{j_2j_3} + \alpha^{\nu}_{j_2i}\varphi^{i}_{j_3j_1} + \alpha^{\nu}_{j_3i}\varphi^{i}_{j_1j_2})]\Big\} \qquad (4.3)$$

$$(\lambda_{j_1} + \lambda_{j_2} + \lambda_{j_3} - \lambda_\nu \neq 0;\ \ \nu, j_1, j_2, j_3, = 1, \dots, n)$$

$$\varphi^{\nu}_{j_1j_2j_3} = a^{\nu}_{j_1j_2j_3} + \frac{2}{3} \sum_{i=1}^{n} [a^{\nu}_{j_1i}\alpha^{i}_{j_2j_3} + a^{\nu}_{j_2i}\alpha^{i}_{j_3j_1} + a^{\nu}_{j_3i}\alpha^{i}_{j_1j_2} - (\alpha^{\nu}_{j_1i}\varphi^{i}_{j_2j_3} + \alpha^{\nu}_{j_2i}\varphi^{i}_{j_3j_1} + \alpha^{\nu}_{j_3i}\varphi^{i}_{j_1j_2})] \qquad (4.4)$$

$$(\lambda_{j_1} + \lambda_{j_2} + \lambda_{j_3} - \lambda_\nu = 0;\ \ \nu, j_1, j_2, j_3 = 1, \dots, n)$$

We emphasize that here too, by virtue of the definition of normal form (1.3), we have

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*) Editor's Note. Obvious misprint in the Russian original; the sentence should read as follows: "...the fraction preceding $B^{\nu}_{j_1\dots j_k}$ in formula (3.2)..."

$\varphi^{\nu}_{j_1j_2j_3} = 0$ when $\lambda_{j_1} + \lambda_{j_2} + \lambda_{j_3} - \lambda_{\nu} \neq 0$. When $\lambda_{j_1} + \lambda_{j_2} + \lambda_{j_3} - \lambda_{\nu} = 0$ the coefficients $\alpha^{\nu}_{j_1j_2j_3}$ of normalizing transformation (1.2) can be chosen arbitrarily.

Note. Let us show that if all the arbitrary quadratic coefficients in (1.2) chosen as zero, i.e. if $\alpha^{\nu}_{j_1j_2} = 0$ when $\lambda_{j_1} + \lambda_{j_2} - \lambda_{\nu} = 0$, then all the summands in the parentheses in (4.4) equal zero. For example, let us show that $\alpha^{\nu}_{j_1i}\varphi^{i}_{j_2j_3} = 0$ $(i = 1 \ldots, n)$. Let us assume at first that $\Delta^{i}_{j_2j_3} = 0$ (see (3.1)), then from (3.4) it follows that $\varphi^{i}_{j_2j_3} = 0$ for these values of i, j_2, j_3, and our assertion is valid. It remains to examine those values of i, j_2, j_3 for which $\Delta^{i}_{j_2j_3} = 1$, i.e. $\lambda_i = \lambda_{j_2} + \lambda_{j_3}$.

From (4.4) we have $\lambda_{\nu} = \lambda_{j_1} + \lambda_{j_2} + \lambda_{j_3}$. By subtracting this from the equality just preceding, we obtain $\lambda_{j_1} + \lambda_i - \lambda_{\nu} = 0$ and, by virtue of the stipulated choice, we have $\alpha^{\nu}_{j_1i} = 0$, i.e. again $\alpha^{\nu}_{j_1i}\varphi^{i}_{j_2j_3} = 0$. The proof is analogous for the remaining summands in the parentheses in (4.4) because they are obtained from the first by a cyclic permutation of the indices j_1, j_2, j_3. Thus, if all the arbitrary quadratic coefficients of the normalizing transformation (1.2) are chosen to be zero, i.e. if

$$\alpha^{\nu}_{j_1j_2} = 0 \quad (\lambda_{j_1} + \lambda_{j_2} - \lambda_{\nu} = 0;\ \nu, j_1, j_2 = 1, \ldots, n)$$

or if quadratic terms are absent in normal form (1.3), then formula (4.4) simplifies to

$$\varphi^{\nu}_{j_1j_2j_3} = a^{\nu}_{j_1j_2j_3} + \frac{2}{3}\sum_{i=1}^{n} [a^{\nu}_{j_1i}\alpha^{i}_{j_2j_3} + a^{\nu}_{j_2i}\alpha^{i}_{j_3j_1} + a^{\nu}_{j_3i}\alpha^{i}_{j_1j_2}] \tag{4.5}$$

$$(\lambda_{j_1} + \lambda_{j_2} + \lambda_{j_3} - \lambda_{\nu} = 0;\ \nu, j_1, j_2, j_3 = 1, \ldots, n)$$

Formulas (4.3) – (4.5) refine formulas (2.4) – (2.6) in [4].

5. Formulas for the coefficients of the fourth powers. For $k=4$ formula (3.3) yields

$$B^{\nu}_{j_1j_2j_3j_4} = a^{\nu}_{j_1j_2j_3j_4} + \frac{1}{2}\sum_{i=1}^{n} (Sa^{\nu}_{j_1i}\alpha^{i}_{j_2j_3j_4} - S\alpha^{\nu}_{j_1i}\varphi^{i}_{j_2j_3j_4} +$$

$$Sa^{\nu}_{j_1j_2i}\alpha^{i}_{j_3j_4} - S\alpha^{\nu}_{j_1j_2i}\varphi^{i}_{j_3j_4}) + \frac{1}{6}\sum_{i,\,h=1}^{n} S\alpha^{i}_{j_1j_2}\alpha^{h}_{j_3j_4}a^{\nu}_{ih}$$

Here S denotes the sum over all combinations of the indices on j in the first factor from among the numbers 1, 2, 3, 4. For the first two sums this reduces to a cyclic permutation of the indices j_1, j_2, j_3, j_4, while for the remaining sums the indices j_1j_2 in the first factors are replaced successively by $j_1j_3, j_1j_4, j_2j_3, j_2j_4, j_3j_4$. By formula (3.2), for the symmetrized coefficients of normalizing transformation (1.2) we have

$$\alpha^{\nu}_{j_1j_2j_3j_4} = \frac{1 - \Delta^{\nu}_{j_1j_2j_3j_4}}{\lambda_{j_1} + \lambda_{j_2} + \lambda_{j_3} + \lambda_{j_4} - \lambda_{\nu}} B^{\nu}_{j_1j_2j_3j_4}$$

$$(\nu, j_1, j_2, j_3, j_4 = 1, \ldots, n)$$

where the $\Delta^{\nu}_{j_1j_2j_3j_4}$ have been defined in (3.1). When $\lambda_{j_1} + \lambda_{j_2} + \lambda_{j_3} + \lambda_{j_4} - \lambda_{\nu} = 0$ the corresponding $\alpha^{\nu}_{j_1j_2j_3j_4}$ can be chosen arbitrarily. Finally, formula (3.4) yields the symmetrized coefficients of normal form (1.6)

$$\varphi^{\nu}_{j_1j_2j_3j_4} = \Delta^{\nu}_{j_1j_2j_3j_4}B^{\nu}_{j_1j_2j_3j_4} \quad (\nu, j_1, j_2, j_3, j_4 = 1, \ldots, n)$$

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APPROXIMATE SOLUTION OF BELLMAN'S EQUATION FOR A CLASS OF OPTIMAL TERMINAL STATE CONTROL PROBLEMS

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We consider the problem of the optimal control of the terminal state of a linear system containing random perturbations in the form of Gaussian white noise. We propose a method for the approximate solution of Bellman's equation for one class of such systems in the case when the solution of the deterministic Bellman equation has discontinuities of the first kind in its values or in the values of its derivatives. As an application of the results obtained we give an approximate solution of Bellman's equation corresponding to one model problem in the control of entry into the atmosphere (see [1, 2]) and we compare the result obtained with the results of the numerical calculations in [2]. Some methods for the approximate solution of Bellman's equation have been studied earlier, for example, in [3 – 6]. Asymptotic expansions with respect to a small parameter, being the noise intensity, were constructed in [4 – 6] for the case when the deterministic Bellman equation corresponding to a system without random perturbations has a smooth solution. Exact solutions of Bellman's equation were obtained in [3] in certain cases when the system has a dimension of one.

1. Statement of the problem. Bellman's equation. Let the equation describing the motion of a system have the form

$$dx/dt = a(x, y, t) + b(x, y, t)u \tag{1.1}$$

Here $0 \leqslant t \leqslant T$, x is a scalar, u is the control function taking values in a convex closed set, $|u(t)| \leqslant p(t)$, $y = (y_1, \ldots, y_n)$ is a vector-valued function satisfying